Benha University
Faculty of Engineering – Shoubra
Mechanical engineering Department

Engineering Mathematics (1) A
Final Exam. **1/1/2018**
Time allowed **3 hour**

First year (production Branch)

Total Mark (105) 21 for each question

Answer the following questions:

Question (1)

(a) Use the power series to solve the differential equation $\boldsymbol{y'' - 2xy' + y = 0}$.

(b) Given $\boldsymbol{w = f(x, y)}$, $\boldsymbol{x = r\cos\theta}$, $\boldsymbol{y = r\sin\theta}$

Show that $(w_x)^2 + (w_y)^2 = (w_r)^2 + \frac{1}{r^2}(w_\theta)^2$

(c) Find the local extrema of the function $f(x, y) = x^2 + 4y^2 - x + 2y$

Answer (a)

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$

We can differentiate the power series term by term, so

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substitute in the equation $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

Let the first series start from 0

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Separate first term from the first and third series to both start from 1

$$2a_2 + a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

Now collect the series $2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (2n-1)a_n\right] x^n = 0$

Then we have $2a_2 + a_0 = 0 \rightarrow a_2 = -\frac{a_0}{2}$

$\left[(n+2)(n+1)a_{n+2}-(2n-1)a_n\right],\ n=1,2,3,\ldots$

$$a_{n+2}=\frac{2n-1}{(n+1)(n+2)}a_n,\ n=1,2,3,\ldots \qquad (7)$$

We solve this recursion relation by putting successively in Equation 7

Put $n=1$: $a_3=\frac{1}{2.3}a_1$

Put $n=2$: $a_4=\frac{3}{3.4}a_2=-\frac{3}{1.2.3.4}=\frac{3}{4!}a_0$

Put $n=3$: $a_5=\frac{5}{4.5}a_3=\frac{5}{4.5}\frac{1}{2.3}a_1=\frac{5}{5!}a_1$

Put $n=4$: $a_6=\frac{7}{5.6}a_4=\frac{3.7}{5.6.4!}a_0=-\frac{3.7}{6!}a_0$

Put $n=5$: $a_7=\frac{1.5.9}{7!}a_1$

Put $n=6$: $a_8=\frac{11}{7.8}a_6=-\frac{3.7.11}{8!}a_0$

Put $n=7$: $a_9=\frac{13}{8.9}a_7=-\frac{1.5.9.13}{9!}a_1$

In general, the even coefficients are given by $a_{2n}=-\frac{3.7.11\ldots(4n-5)}{\left(2n\right)!}a_0$

And the odd coefficients are given by $a_{2n-1}=\frac{1.5.9\ldots(4n-3)}{\left(2n+1\right)!}a_1$

The solution is $y=a_0\left(1-\frac{1}{2!}x^2-\frac{3}{4!}x^4-\frac{3.7}{6!}x^6-\frac{3.7.11}{8!}x^8+\ldots\right)$

$$+a_1\left(x+\frac{1}{3!}x^3+\frac{1.5}{5!}x^5+\frac{1.5.9}{7!}x^7+\frac{1.5.9.13}{9!}x^9+\ldots\right)$$

or $y=a_0\left(1-\frac{1}{2!}x^2-\sum_{n=2}^{\infty}\frac{3.7.11\ldots(4n-5)}{(2n)!}x^{2n}\right)$

$$+a_1\left(x+\sum_{n=1}^{\infty}\frac{1.5.9.13\ldots(4n-3)}{(2n+1)!}x^{2n+1}\right).$$

Answer (b)

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial w}{\partial x}\cos\theta+\frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial w}{\partial x}(-r\sin\theta)+\frac{\partial w}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial w}{\partial r}\right)^2=\left(\frac{\partial w}{\partial x}\cos\theta+\frac{\partial w}{\partial y}\sin\theta\right)^2$$

$$=\left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta+2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\cos\theta\sin\theta+\left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial w}{\partial x}(-r\ \sin\theta)+\frac{\partial w}{\partial y}(r\ \cos\theta)\right)^2$$

$$=\left(\frac{\partial w}{\partial x}\right)^2 r^2\sin^2\theta-2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}r^2\sin\theta\cos\theta+\left(\frac{\partial w}{\partial y}\right)^2 r^2\cos^2\theta$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2\frac{1}{r^2}=\left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta-2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta+\left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta$$

$$\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{\partial w}{\partial \theta}\right)^2\frac{1}{r^2}=\left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta+\cancel{2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\cos\theta\sin\theta}+\left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta$$

$$+\left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta-\cancel{2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta}+\left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta$$

$$=\left(\frac{\partial w}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta)+\left(\frac{\partial w}{\partial y}\right)^2(\cos^2\theta+\sin^2\theta)=\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2$$

Answer (c)

$$f(x,y)=x^2+4y^2-x+2y$$

$$\frac{\partial f}{\partial x}=2x-1,\qquad \frac{\partial f}{\partial y}=8y+2$$

Since f_x and f_y exist for every (x,y) the only critical points are the solution of the following system of two equation in two variables

$$\frac{\partial f}{\partial x}=2x-1=0\qquad \frac{\partial f}{\partial y}=8y+2=0$$

Which is the point $(\frac{1}{2},\frac{-1}{4})$

$$\frac{\partial^2 f}{\partial x^2}=2,\quad \frac{\partial^2 f}{\partial y^2}=8,\quad \frac{\partial^2 f}{\partial y\,\partial x}=0 \qquad D(x,y)=\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left[\frac{\partial^2 f}{\partial y\,\partial x}\right]^2=(2)(8)-(0)^2=16>0$$

Since $\frac{\partial^2 f}{\partial x^2}=2>0$ then $f(\frac{1}{2},\frac{-1}{4})$ is a local minimum for the function $f(x,y)$

Question (2)

(a) If $\phi=2x^3y^2z^4$ find $\vec{\nabla}.\vec{\nabla}\phi$. and $\vec{\nabla}\times\vec{\nabla}\phi$

(b) Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}=(5xy-6x^2)\vec{i}+(2y-4x)\vec{j}$ along the curve $y=x^3$ from the point $(1,1)$ to the point $(2,8)$.

(c) Evaluate $\int_0^2\int_{x^2}^{2x}(x^3+4y)\,dy\,dx$

Answer (a)

$$\vec{\nabla}\phi=\vec{\nabla}(2x^3y^2z^4)=\frac{\partial}{\partial x}(2x^3y^2z^4)\vec{i}+\frac{\partial}{\partial y}(2x^3y^2z^4)\vec{j}+\frac{\partial}{\partial z}(2x^3y^2z^4)\vec{k}$$

$$=6x^2y^2z^4\vec{i}+4x^3yz^4\vec{j}+8x^3y^2z^3\vec{k}$$

$$\vec{\nabla}.\vec{\nabla}\phi=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}).(6x^2y^2z^4\vec{i}+4x^3y\ z^4\vec{j}+8x^3y^2z^3\vec{k})$$

$$=\frac{\partial}{\partial x}6x^2y^2z^4+\frac{\partial}{\partial y}4x^3y\ z^4+\frac{\partial}{\partial z}8x^3y^2z^3$$

$$=12xy^2z^4+4x^3z^4+24x^3y^2z^2.$$

Answer (b)

$$\therefore\int_C \vec{F}.d\vec{r}=\int_C\left[(5xy-6x^2)\vec{i}+(2y-4x)\vec{j}\right].(dx\,\vec{i}+dy\,\vec{j})$$

$$=\int_C\left[(5xy-6x^2)dx+(2y-4x)dy\right]$$

Substitute by $y=x^3$ and $dy=3x^2dx$

to determine the limits of integration we use the start and end points as follows. at $(1,1)$ $x=1$, at $(2,8)$ $x=2$

$$\therefore \int_C \left[(5xy - 6x^2)dx + (2y - 4x)dy\right]$$

$$= \int_1^2 \left[5x\left(x^3\right) - 6x^2 + \left(2(x^3) - 4x\right)\left(3x^2\right)\right]dx$$

$$= \int_1^2 \left[6x^5 + 5x^4 - 12x^3 - 6x^2\right]dx = \left[x^6 + x^5 - 3x^4 - 2x^3\right]_1^2$$

$$= (2^6 - 1) + (2^5 - 1) - 3(2^4 - 1) - 2(2^3 - 1)$$

$$= 65 + 31 - 45 - 14 = 35$$

Answer (c)

$$\int_0^2 \int_{x^2}^{2x} (x^3 + 4y)dy\,dx = \int_0^2 \left[\int_{x^2}^{2x} (x^3 + 4y)dy\right]dx = \int_0^2 (x^3 y + 2y^2)\Big|_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[(2x^4 + 8x^2) - (x^5 + 2x^4)\right]dx = \int_0^2 \left[8x^2 - x^5\right]dx$$

$$= \frac{8}{3}x^3 - \frac{1}{6}x^6\Big|_{x=0}^{x=2} = \frac{64}{3} - \frac{64}{6} = \frac{64}{6} = \frac{32}{3}$$

Question (3)

Find the general solution of the following differential equations:

(a) $xydx + (x^2 + 1)dy = 0$

(b) $(3x + y)dx + (x + 3y)dy = 0$

(c) $y' + y.\tan x = \sin x$

Answer (a)

Divided the equation by $y(x^2 + 1)$

$\dfrac{x}{(x^2+1)}dx + \dfrac{dy}{y} = 0$ The variables are separated, integrate the equation

$$\int \frac{x}{(x^2+1)}dx + \int \frac{dy}{y} = c$$

$$\frac{1}{2}\ln(x^2+1) + \ln y = \ln c$$

$$\ln(x^2+1)^{\frac{1}{2}} + \ln y = \ln c$$

$$\ln\sqrt{(x^2+1)} + \ln y = c \rightarrow \ln y\sqrt{(x^2+1)} = \ln c \rightarrow y\sqrt{(x^2+1)} = c$$

The solution of the equation is $\boxed{y^2x^2 + y^2 = C}$

Another solution

$$xydx + (x^2+1)dy = 0$$

$$xydx + (x^2dy + dy) = 0$$

$$x(ydx + xdy) + dy = 0$$

$$xd(xy) + dy = 0$$

$$(xy)d(xy) + ydy = 0$$

Integrate $\frac{1}{2}(xy)^2 + \frac{1}{2}y^2 = c \Rightarrow \boxed{x^2y^2 + y^2 = C}$

Answer (b)

$$M = (3x+y), \qquad N = (x+3y)$$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 1$$

now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ thus the equation is exact differential equation

to find the solution function $F(x,y)$ we use (4) since $\frac{\partial F(x,y)}{\partial x} = M(x,y)$

$\therefore F(x,y) = \int M(x,y)dx = \int (3x+y)dx = \frac{3}{2}x^2 + yx + f(y)$ **(*)** we set the constant of integration $f(y)$ as a function of y

to determine $f(y)$ differentiate (*) with respect to y

$$\frac{\partial F(x,y)}{\partial y} = x + f'(y) \qquad \textbf{(**)}$$

using (5) i.e. equate (**) by $N(x,y)$

$$\because N(x,y)=\frac{\partial F(x,y)}{\partial y}$$

$$\therefore x+3y=x+f'(y)$$

$f'(y)=3y$.which show that $f(y)=\frac{3}{2}y^2$

Hence the general solution is $F(x,y)=\frac{3}{2}x^2+yx+\frac{3}{2}y^2=C$

we can use the direct method $F(x,y)=\int_0^x M(x,y)dx+\int_0^y N(0,y)dy=C$

$$F(x,y)=\int_0^x (3x+y)dx+\int_0^y 3ydy=C$$

$$\boxed{F(x,y)=\frac{3}{2}x^2+yx+\frac{3}{2}y^2=C}$$

We can solve the equation as homogenous equation.

Another solution

$$(3x+y)dx+(x+3y)dy=0$$

$$3(xdx+dy)+(ydx+xdy)=0$$

$$\frac{3}{2}d\left(x^2+y^2\right)+d\left(xy\right)=0$$

Integrate the last equation then $\frac{3}{2}d\left(x^2+y^2\right)+d\left(xy\right)=0$

$$\boxed{\frac{3}{2}\left(x^2+y^2\right)+\left(xy\right)=C}$$

Answer (c)

The equation is a linear and $P(x)=\tan x,\quad Q(x)=\sin x$

We can determine the integrating factor

$$\int P(x)dx=\int \tan xdx=\ln\sec x$$

$$\therefore \mu=e^{\ln\sec x}=\sec x$$

multiply the equation by $\sec x$

$y'\sec x + y\sec x\tan x = \sec x\sin x = \tan x$

which is exact differential equation

and the left side is the derivative of $y\sec x$

$\therefore d(y\sec x) = \tan x$ by integration

$y\sec x = \int \tan x dx = \sec x + C$

$\boxed{y\sec x = \sec x + C}$

Question (4)

(a) Find the general solution for $y'' - 5y' - 6y = e^x \sinh 6x$

(b) By variation of parameter solve $y'' + n^2 = \operatorname{cosec} nx$.

(c) Find the general solution for $xy'' + (x-1)y' - y = 0$ given that $y = e^{-x}$ is one solution.

Answer (a)

The characteristic equation is

$m^2 - 5m + 6 = 0$ then $(m-2)(m-3) = 0 \to m = 2,3$ and

$y_c = C_1 e^{3x} + C_2 e^{2x}$

and

$$y_p = \frac{1}{D^2 - 5D + 6} e^x \sinh 6x = e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \sinh 2x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 5D - 5 + 6} \sinh 6x = e^x \frac{1}{D^2 - 3D + 2} \sinh 6x$$

$$= e^x \frac{1}{6^2 - 3D + 2} \sinh 6x = e^x \frac{1}{38 - 3D} \sinh 6x = e^x \frac{(38 - 3D)}{38^2 - 9D^2} \sinh 6x$$

$$= \frac{e^x}{38^2 - 9(6)^2}(38\sinh 6x - 18\cosh 6x)$$

Then the general solution in the form

$$\boxed{y_G = C_1 e^{3x} + C_2 e^{2x} + \frac{e^x}{38^2 - 9(6)^2}(38\sinh 6x - 18\cosh 6x)}$$

Another solution

$$y_p = \frac{1}{D^2-5D+6}e^x \sinh 6x = \frac{1}{2}\frac{1}{D^2-5D+6}e^x\left(e^{6x}-e^{-6x}\right)$$

$$= \frac{1}{2}\frac{1}{D^2-5D+6}\left(e^{7x}-e^{-5x}\right) = \frac{1}{2}\left(\frac{1}{D^2-5D+6}e^{7x} - \frac{1}{D^2-5D+6}e^{-5x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{7^2-5(7)+6}e^{7x} - \frac{1}{(-5)^2-5(5)+6}e^{-5x}\right) = \frac{1}{2}\left(\frac{1}{20}e^{7x} - \frac{1}{6}e^{-5x}\right)$$

Then the general solution in the form

$$\boxed{y_G = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{2}\left(\frac{1}{20}e^{7x} - \frac{1}{6}e^{-5x}\right)}$$

Answer (b)

The characteristic equation for the homogeneous equation in the form $m^2+n^2=0$ which has a roots are $m=\pm ni$ then the solution

$$y_c = A\cos nx + B\sin nx$$

Where A, B are arbitrary constant

Let the general solution in the form
$y_G = A(x)\cos nx + B(x)\sin nx$ (3)

Subject to $A'\cos nx + B'\sin nx = 0$ (4)

Then $y' = -nA\sin nx + nB\cos nx$ (5)

$$y'' = -nA'\sin nx - n^2 A\cos nx + nB'\cos nx - n^2 B\sin nx$$

Substitute in (1) then $-nA'\sin nx + nB'\cos nx = \operatorname{cosec} nx$ (6)

Solve (4) with (6)-multiply (4) by $n\sin nx$ and (6) by $\cos nx$ we have

$nA'\cos nx\sin nx + nB'\sin^2 nx = 0$

$-nA'\sin nx\cos nx + nB'\cos^2 nx = \cos nx\ \operatorname{cosec} nx$

Adding the equations then $nB'(\sin^2 nx + \cos^2 nx) = \cot nx$

$B' = \frac{1}{n}\cot nx \quad \rightarrow \quad B(x) = \frac{1}{n^2}\ln\sin nx + C_2$

Use equation (4) $A'\cos nx + \frac{1}{n}\cot nx\sin nx = 0$

$A' = -\frac{1}{n} \quad \rightarrow \quad A = \frac{-x}{n} + C_1$

Substitute in (3) $y_G = \left(\frac{-x}{n} + C_1\right)\cos nx + \left(\frac{1}{n^2}\ln\sin nx + C_2\right)\sin nx$

$$\boxed{y_G = C_1\cos nx + C_2\sin nx + \frac{\sin nx}{n^2}\ln\sin nx + \frac{x}{n}\cos nx}$$

Answer (c)

$$xy'' + (x-1)y' - y = 0$$

$y = e^{-x}$ is a solution for the equation

Let the general is $y = v\,e^{x}$

$$y' = v'e^{-x} - ve^{-x}$$

$$y'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

Substitute in the homogeneous equation

$$xy'' + (x-1)y' - y = 0$$

$$v''xe^{-x} - 2v'xe^{-x} + vxe^{-x} + (x-1)(v'e^{-x} - ve^{-x}) - ve^{-x} = 0$$

$$v''x - v'x - v' = 0$$

$$\frac{v''}{v'} = \frac{x+1}{x} \Rightarrow \ln v' = x + \ln x \Rightarrow v' = c_1xe^{x} \Rightarrow v = c_1\left(xe^{x} - e^{x}\right) + c_2$$

$$\boxed{y = c_1(x-1) + c_2e^{-x}}$$

Question (5)

(a) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (2x - 3y)\,\vec{i} + y\,\vec{j}$ and C is the circle $x^2 + y^2 = 4$.

(b) Apply Green's theorem to evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (x - y)\,\vec{i} + (x + y)\,\vec{j}$ and C is the closed curve in xy – plane consisting of $y = x^2, x = y^2$.

(c) Use the divergence theorem to evaluate $\oiint_S \vec{F}.\vec{n}ds$

where $\vec{F} = 2xy\,\vec{i} + y\,z^2\,\vec{j} + xz\,\vec{k}$ and S is the surface of parallelogram bounded by $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3.$.

Answer (a)

by using parametric equation of the circle $x = 2\cos\theta, \qquad y = 2\sin\theta$

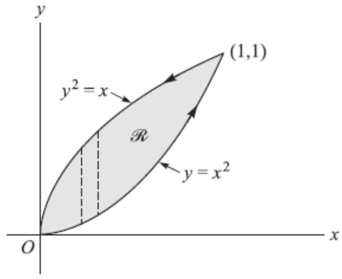
$$\therefore dx = -2\sin\theta d\theta, \quad dy = 2\cos\theta d\theta, \quad 0 \le \theta \le 2\pi$$

$$\therefore W = \int_C \vec{F}.d\vec{r} = \int_C (2x - 3y)dx + ydy$$

$$= \int_0^{2\pi} [(4\cos\theta) - 6\sin\theta][-2\sin\theta]d\theta + [2\sin\theta][2\cos\theta]d\theta$$

$$= \int_0^{2\pi} (-4\sin\theta\cos\theta + 12\sin^2\theta)d\theta = 12\pi$$

Answer (b)



The two curves intersects at $(0,0),(1,1)$

$$\int_C \vec{F}.d\vec{r} = \int_C \left[(x-y)\vec{i} + (x+y)\vec{j}\right].\left[dx\,\vec{i} + dy\,\vec{j}\right]$$

$$= \int_C \left[(x-y)\,dx + (x+y)\,dy\right] = \iint \left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\right)dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2dxdy = 2\int_0^1 \left(\sqrt{x} - x^2\right)dx = 2\left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$$

Answer (c)

$$\therefore \iiint_V \vec{\nabla}.\vec{F}\,dV = \iint_S \vec{F}.\vec{n}ds$$

$$\vec{F} = 2yx\,\vec{i} + yz^2\vec{j} + xz\,\vec{k} \quad and \quad \vec{\nabla}.\vec{F} = 2y + z^2 + x$$

$$\iint_S \vec{F}.\vec{n}ds = \iiint_V \vec{\nabla}.\vec{F}\,dV = \iiint_V \left[2y + z^2 + x\right]dxdydz$$

$$= \int_0^3 \int_0^1 \int_0^2 \left[2y + z^2 + x\right]dxdydz = \int_0^3 \int_0^1 \left[2yx + z^2x + \frac{1}{2}x^2\right]_0^2 dydz$$

$$= \int_0^3 \int_0^1 \left[4y + 2z^2 + 2\right]dydz = \int_0^3 \left[2y^2 + 2z^2y + 2y\right]_0^1 dz$$

$$= \int_0^3 \left[4 + 2z^2\right]dz = \left[4z + \frac{2}{3}z^3\right]_0^3 = 12 + 18 = 30$$
